

Transfer of stable equivalences of Morita type

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Abstract

Let A and B be finite-dimensional k -algebras over a field k such that $A/\text{rad}(A)$ and $B/\text{rad}(B)$ are separable. In this note, we consider how to transfer a stable equivalence of Morita type between A and B to that between eAe and fBf , where e and f are idempotent elements in A and in B , respectively. In particular, if the Auslander algebras of two representation-finite algebras A and B are stably equivalent of Morita type, then A and B themselves are stably equivalent of Morita type. Thus, combining a result with Liu and Xi, we see that two representation-finite algebras A and B over a perfect field are stably equivalent of Morita type if and only if their Auslander algebras are stably equivalent of Morita type. Moreover, since stable equivalence of Morita type preserves n -cluster tilting modules, we extend this result to n -representation-finite algebras and n -Auslander algebras studied by Iyama.

2000 AMS Classification: 16G10, 18G05, 16S50; 16P10, 20C05, 18E10.

Key words: stable equivalence of Morita type, Auslander algebra, n -cluster tilting module.

1 Introduction

Stable equivalence of Morita type introduced by Broué in [2] is one of the fundamental equivalence relations for algebras and groups. It is of considerable interest in the modular representation theory of finite groups, or more generally, of finite-dimensional self-injective algebras. The notion is intimately related to the celebrated conjecture of Broué, which says that certain blocks of group algebras with abelian defect groups should be derived-equivalent (see [2] and [12] for precise formulation); the connection can be seen from Rickard's result that a derived equivalence between self-injective algebras induces a stable equivalence of Morita type [13]. Recently, it is shown that stable equivalence of Morita type is also of particular interest for general finite-dimensional algebras, for example, it preserves representation dimension [15], representation type [7], Hochschild homological and cohomological groups [10, 14], and the absolute value of Cartan determinant [14]. As is known, stable equivalence of Morita type occurs frequently not only in the block theory of finite groups [8], but also in the representation theory of general finite-dimensional algebras. A plenty of such examples are constructed in [9, 10, 11].

Moreover, it is shown in [10] that, for two finite-dimensional k -algebras A and B over a field k of finite representation-type, if A and B are stably equivalent of Morita type, then their Auslander algebras are also stably equivalent of Morita type. A natural question is whether the converse of this statement is true.

In this note, we shall consider the general question of how to transfer a stable equivalence of Morita type between two algebras A and B over a field to that between eAe and fBf , where e and f are idempotent elements in A and B , respectively. We say that two bimodules ${}_AM_B$ and

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${}_B N_A$ define a stable equivalence of Morita type between A and B if M and N are projective as one-sided modules, and there are a projective A - A -bimodule P and a projective B - B -bimodule Q such that $M \otimes_B N \simeq A \oplus P$ and $N \otimes_A M \simeq B \oplus Q$ as bimodules. With these notations in mind, our main result reads as follows:

Theorem 1.1 *Suppose that A and B are finite-dimensional k -algebras over a field k such that both $A/\text{rad}(A)$ and $B/\text{rad}(B)$ are separable. Let ${}_A M_B$ and ${}_B N_A$ be two bimodules defining a stable equivalence of Morita type between A and B . If $e^2 = e \in A$ such that $Pe \in \text{add}(Ae)$, and if f is a sum of the pairwise orthogonal primitive idempotent elements $f_j \in B$ in the decomposition $Ne \simeq \bigoplus_{j=1}^m (Bf_j)^{r_j}$, then the bimodules eMf and fNe define a stable equivalence of Morita type between eAe and fBf .*

From this result, we have the following corollary which supplies a positive answer to our previous question on Auslander algebras. For the unexplained notion in the corollary, we refer the reader to Section 3 below.

Corollary 1.2 *Suppose that A and B are finite-dimensional k -algebras over a perfect field k . Assume that A and B are n -representation-finite. If the n -Auslander algebras of A and B are stably equivalent of Morita type, then A and B themselves are stably equivalent of Morita type.*

The proof of Theorem 1.1 is given in the next section.

2 Proof of the main result

Throughout this note, k denotes a fixed field. Given a finite-dimensional k -algebra A , we denote by $A\text{-mod}$ the category of all finitely generated left A -modules. If $M \in A\text{-mod}$, we denote by $\text{add}(M)$ the full subcategory of $A\text{-mod}$ consisting of all modules X which are direct summands of finite sums of copies of M . By an algebra we mean a finite-dimensional k -algebra, and by a module we mean a left module. The global and dominant dimensions of an algebra A are denoted by $\text{gl.dim}(A)$ and $\text{dom.dim}(A)$, respectively. The composition of two homomorphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ is denoted by $fg : X \rightarrow Z$, and the usual k -duality is denoted by $D := \text{Hom}_k(-, k)$. Let us recall the definition of a stable equivalence of Morita type.

Definition 2.1 [2] *Suppose that A and B are two (arbitrary) k -algebras. We say that A and B are stably equivalent of Morita type if there exist an A - B -bimodule ${}_A M_B$ and a B - A -bimodule ${}_B N_A$ such that*

- (1) M and N are projective as one-sided modules, and
- (2) $M \otimes N \simeq A \oplus P$ as A - A -bimodules for some projective A - A -bimodule P , and $N \otimes M \simeq B \oplus Q$ as B - B -bimodules for some projective B - B -bimodule Q .

Note that if A and B are stably equivalent of Morita type, then their opposite algebras A^{op} and B^{op} are also stably equivalent of Morita type.

Let A be a representation-finite algebra. An A -module X is called an additive generator for $A\text{-mod}$ if $\text{add}(X) = A\text{-mod}$, that is, every indecomposable A -module is isomorphic to a direct summand of X . Let X be an additive generator for $A\text{-mod}$. The endomorphism algebra $\Lambda = \text{End}_A(X)$ of X is called the Auslander algebra of A . (This is unique up to Morita equivalence.) Auslander algebras can be described by two homological properties: An algebra A is an Auslander algebra if (1) $\text{gl.dim}(A) \leq 2$; and (2) if $0 \rightarrow A \rightarrow I_0 \rightarrow I_1 \rightarrow I_2 \rightarrow 0$ is a minimal injective resolution of A , then I_0 and I_1 are projective.

An A -module $X \in A\text{-mod}$ is called a generator for $A\text{-mod}$ if $\text{add}({}_A A) \subseteq \text{add}(X)$; a cogenerator for $A\text{-mod}$ if $\text{add}(D({}_A A)) \subseteq \text{add}(X)$, and a generator-cogenerator if it is both a generator and a cogenerator for $A\text{-mod}$. Clearly, for a representation-finite algebra A , an additive generator for $A\text{-mod}$ is a generator-cogenerator for $A\text{-mod}$.

In the following, we shall introduce some notations. Assume that A is a k -algebra.

Let T be an arbitrary A -module, and suppose B is the endomorphism algebra of T . We consider the following subcategories related to T .

$Gen({}_A T) = \{X \in A\text{-mod} \mid \text{there is a surjective homomorphism from } T^m \text{ to } X \text{ with } m \geq 1\}.$
 $Pre({}_A T) = \{X \in A\text{-mod} \mid \text{there is an exact sequence } T_1 \rightarrow T_0 \rightarrow X \text{ with all } T_i \in \text{add}({}_A T)\}.$
 $App({}_A T) = \{X \in A\text{-mod} \mid \text{there is a homomorphism } g : T_0 \rightarrow X \text{ with } T_0 \in \text{add}({}_A T) \text{ such that } \text{Ker}(g) \in Gen({}_A T) \text{ and } g \text{ is a right } \text{add}({}_A T)\text{-approximation of } X\}.$

The following lemma is known, for a proof, we refer to [16, Lemma 2.1].

Lemma 2.2 *Let X be an arbitrary A -module. Recall that $B = \text{End}_A(T)$ and ${}_A T_B$ is a natural bimodule. Then:*

- (1) *Let Y be a right B -module. The natural homomorphism $\delta: Y \otimes_B \text{Hom}_A(T, X) \rightarrow \text{Hom}_B(\text{Hom}_A(X, T), Y)$, given by $y \otimes f \mapsto \delta_{y \otimes f}$ with $\delta_{y \otimes f}(g) = y(fg)$ for $y \in Y, f \in \text{Hom}_A(T, X), g \in \text{Hom}_A(X, T)$, is an isomorphism if $X \in \text{add}({}_A T)$.*
- (2) *If $X' \in \text{add}({}_A T)$, or $X \in \text{add}({}_A T)$, then the composition map $\mu: \text{Hom}_A(X', T) \otimes_B \text{Hom}_A(T, X) \rightarrow \text{Hom}_A(X', X)$ given by $f \otimes_B g \mapsto fg$ is bijective.*
- (3) *Let C be a k -algebra, and suppose ${}_A X_C$ is an A - C -bimodule. If ${}_A X \in Gen({}_A T)$, then the evaluation map $e_X: T \otimes_B \text{Hom}_A(T, X) \rightarrow X$ is surjective as A - C -bimodules. If $X \in App({}_A T)$, then e_X is an isomorphism as A - C -bimodules. Conversely, if e_X is bijective as A -modules, then $X \in App({}_A T)$.*

The next lemma is taken from [14, Lemma 2.1].

Lemma 2.3 [14] (1) *Let A, B, C and E be k -algebras, and let ${}_A X_B$ and ${}_B Y_E$ be bimodules with X_B projective. Put $X^* = \text{Hom}_B(X, B)$. Then the natural homomorphism $\phi: {}_A X \otimes_B Y_E \rightarrow \text{Hom}_B({}_B X_A^*, {}_B Y_E)$, defined by $f \mapsto (xf)y$ for $x \in X, y \in Y$ and $f \in X^*$, is an isomorphism of A - E -bimodules, where the image of x under f is denoted by xf .*

(2) *In the situation $({}_E P_A, {}_C X_B, {}_A U_B)$, if P_A is projective, or if X_B is projective, then ${}_E P \otimes_A \text{Hom}_B({}_C X_B, {}_A U_B) \simeq \text{Hom}_B({}_C X_B, {}_E P \otimes_A U_B)$ as E - C -bimodules. Dually, in the situation $({}_A P_E, {}_B X_C, {}_B U_A)$, if ${}_A P$ is projective, or if ${}_B X$ is projective, then $\text{Hom}_B({}_B X_C, {}_B U_A) \otimes_A P_E \simeq \text{Hom}_B({}_B X_C, {}_B U \otimes_A P_E)$ as C - E -bimodules.*

The following is a well-known result due to Auslander (for example, see [1, Proposition 5.6, p.214]).

Lemma 2.4 *Let Λ be an Artin algebra such that $\text{gl.dim}(\Lambda) \leq 2 \leq \text{dom.dim}(\Lambda)$. Let U be a Λ -module such that $\text{add}(U)$ is the full subcategory of $\Lambda\text{-mod}$ consisting of all projective-injective Λ -modules. Then*

- (1) $A := \text{End}_\Lambda(U)$ is representation-finite.
- (2) Λ is Morita equivalent to $\text{End}_A(X)$, where X is an additive generator for $A\text{-mod}$.

For our proof of Theorem 1.1, we also need the following lemma in [3, Theorem 2.7, Corollary 3.1, Lemma 3.2], see also [14, Lemma 3.3].

Lemma 2.5 [3] *Suppose that A and B are finite-dimensional k -algebras over a field k such that A and B have no separable direct summands and that $A/\text{rad}(A)$ and $B/\text{rad}(B)$ are separable. Assume that ${}_A M_B$ and ${}_B N_A$ are indecomposable bimodules that define a stable equivalence of Morita type between A and B . Then:*

- (1) *There are isomorphisms of bimodule: $N \cong \text{Hom}_A(M, A) \cong \text{Hom}_B(M, B)$ and $M \cong \text{Hom}_A(N, A) \cong \text{Hom}_B(N, B)$.*
- (2) *Both $(N \otimes_A -, M \otimes_B -)$ and $(M \otimes_B -, N \otimes_A -)$ are adjoint pairs of functors.*
- (3) *If ${}_A I$ is injective, then so is $N \otimes_A I$.*

It follows from Lemma 2.5 that the following result is true. Note that the last statement in Lemma 2.6 below follows from [14, Lemma 4.5].

Lemma 2.6 *Suppose that A and B are finite-dimensional k -algebras over a field k such that $A/\text{rad}(A)$ and $B/\text{rad}(B)$ are separable. Assume that $\{e_1, \dots, e_n\}$ and $\{f_1, \dots, f_m\}$ are complete sets of pairwise orthogonal primitive idempotents in A and B , respectively. Let e be the sum of all*

those e_i for which Ae_i is projective-injective, and let f be the sum of all those f_j for which Bf_j is projective-injective. If M and N are indecomposable bimodules that define a stable equivalence of Morita type between A and B , then $Ne \simeq N \otimes_A Ae \in \text{add}(Bf)$, $Mf \simeq M \otimes_B Bf \in \text{add}(Ae)$, and $Pe \in \text{add}(Ae)$.

Proof of Theorem 1.1:

Suppose that A and B are two algebras over a field k such that $A/\text{rad}(A)$ and $B/\text{rad}(B)$ are separable, and suppose that ${}_A M_B$ and ${}_B N_A$ define a stable equivalence of Morita type between A and B . We may assume that both A and B have no separable summands since the direct sum of A with a separable k -algebra is always stably equivalent of Morita type to A itself. Furthermore, by [9, Lemma 2.2], we may assume that M and N are indecomposable as bimodules. Then $(M \otimes_B -, N \otimes_A -)$ and $(N \otimes_A -, M \otimes_B -)$ are adjoint pairs by Lemma 2.5.

To prove Theorem 1.1, we shall show that the bimodules eMf and fNe satisfy the conditions of a stable equivalence of Morita type between eAe and fBf .

(1) fNe is projective as both an fBf -module and a right eAe -module.

In fact, we have $fNe \simeq fB \otimes_B Ne \simeq \text{Hom}_B(Bf, B) \otimes_B Ne \simeq \text{Hom}_B(Bf, {}_B Ne)$ by Lemma 2.3. Since $Ne \in \text{add}(Bf)$ by the definition of f , we see that $\text{Hom}_B(Bf, Ne)$ is projective as an fBf -module, that is, fNe is projective as an fBf -module. To see that fNe is a projective right eAe -module, we notice that $\text{add}(Mf) = \text{add}(M \otimes_B Bf) = \text{add}(M \otimes_B Ne) = \text{add}((M \otimes_B N)e) = \text{add}(Ae \oplus Pe) = \text{add}(Ae)$, here we use the assumption $Pe \in \text{add}(Ae)$. Since $(M \otimes_B -, N \otimes_A -)$ is an adjoint pair, it follows from $\text{Hom}_B(Bf, {}_B N \otimes_A Ae) \simeq \text{Hom}_A(M \otimes_B Bf, Ae) \simeq \text{Hom}_A(Mf, Ae)$ that fNe is projective as a right eAe -module since $Mf \in \text{add}(Ae)$. Thus (1) is proved.

(2) eMf is projective as both an eAe -module and a right fBf -module. The proof of (2) is similar to that of (1), we omit it here.

(3) $fNe \otimes_{eAe} eMf \simeq fBf \oplus fQf$ as bimodules.

Indeed, by the associativity of tensor products, we have the following isomorphisms of fBf - fBf -bimodules:

$$\begin{aligned} fNe \otimes_{eAe} eMf &\simeq fN \otimes_A Ae \otimes_{eAe} eA \otimes_A Mf \\ &\simeq fN \otimes_A Ae \otimes_{eAe} \text{Hom}(Ae, A) \otimes_A Mf \\ &\simeq fN \otimes_A Ae \otimes_{eAe} \text{Hom}(Ae, {}_A Mf) \quad (\text{by Lemma 2.3}) \\ &\simeq fN \otimes_A Mf \quad (\text{by Lemma 2.2}). \end{aligned}$$

Since M and N define the stable equivalence of Morita type between A and B , we have $N \otimes_A M \simeq B \oplus Q$ as B - B -bimodules. This implies that $fNe \otimes_{eAe} eMf \simeq fN \otimes Mf \simeq fB \otimes_B N \otimes_A M \otimes_B Bf \simeq fB \otimes_B (B \oplus Q) \otimes_B Bf \simeq fBf \oplus fB \otimes_B Q \otimes_B Bf \simeq fBf \oplus fQf$.

(4) The bimodule fQf in (3) is projective.

In fact, since Q is a projective B - B -bimodule and since $B/\text{rad}(B)$ is separable, the bimodule Q is isomorphic to a direct sum of modules of the form $Q_1 \otimes_k Q_2$, where Q_1 is a projective left B -module and Q_2 is a projective right B -module. Since $fNe \otimes_{eAe} eMf$ and fBf are projective as left fBf -modules, we infer that $fQ_1 \otimes_k Q_2 f$ is a projective fBf -module. It follows that fQ_1 is a projective fBf -module. Similarly, $Q_2 f$ is a projective right fBf -module. Thus fQf , which is isomorphic to a direct sum of modules of the form $fQ_1 \otimes_k Q_2 f$ with fQ_1 a projective fBf -module and $Q_2 f$ a projective right fBf -module, is projective as an fBf - fBf -bimodule.

(5) Similarly, we can show that $eMf \otimes_{fBf} fNe \simeq eAe \oplus ePe$ and that ePe is a projective eAe - eAe -bimodule.

Indeed, we have

$$\begin{aligned} eMf \otimes_{fBf} fNe &\simeq eM \otimes_B Bf \otimes_{fBf} fB \otimes_B Ne \\ &\simeq eM \otimes_B Bf \otimes_{fBf} \text{Hom}(Bf, B) \otimes_B Ne \\ &\simeq eM \otimes_B Bf \otimes_{fBf} \text{Hom}(Bf, {}_B Ne) \quad (\text{by Lemma 2.3}) \\ &\simeq eM \otimes_B Ne \quad (\text{by Lemma 2.2}). \end{aligned}$$

Since M and N define the stable equivalence of Morita type between A and B , we have $M \otimes_B N \simeq A \oplus P$ as A - A -bimodules. This implies that $eMf \otimes_{fBf} fNe \simeq eM \otimes_A Ne \simeq eAe \oplus ePe$. Now, we show that the bimodule ePe is projective.

In fact, since P is a projective A - A -bimodule and since $A/\text{rad}(A)$ is separable, the bimodule P is isomorphic to a direct sum of modules of the form $P_1 \otimes_k P_2$, where P_1 is a projective left A -module and P_2 is a projective right A -module. Since $eMf \otimes_{fBf} fNe$ and eAe are projective as left eAe -modules, we infer that $eP_1 \otimes_k P_2e$ is a projective eAe -module. It follows that eP_1 is a projective eAe -module. Similarly, P_2e is a projective right eAe -module. Thus ePe , which is isomorphic to a direct sum of modules of the form $eP_1 \otimes_k P_2e$ with eP_1 a projective eAe -module and P_2e a projective right eAe -module, is projective as an eAe - eAe -bimodule.

Thus, by definition, the bimodules eMf and fNe define a stable equivalence of Morita type between eAe and fBf . This finishes the proof of Theorem 1.1. \square

Remarks. (1) In Theorem 1.1, if e is an idempotent element in A such that every indecomposable projective-injective A -module is isomorphic to a summand of Ae , then $Pe \in \text{add}(Ae)$. This follows immediately from the proof of [14, Lemma 4.5]. In fact, under the assumption of Theorem 1.1, we infer that $\nu_A^i({}_A P)$ is projective-injective for all $i \geq 0$, where ν_A is the Nakayama functor $D\text{Hom}_A(-, {}_A A)$. Hence, if e is an idempotent element in A such that every indecomposable projective-injective A -module X with $\nu_A^j X$ projective-injective for all $j \geq 0$ belongs to $\text{add}(Ae)$, then ${}_A P \in \text{add}(Ae)$.

(2) As was pointed out in [3, Section 4], if e is an idempotent in A and if f is an idempotent in B such that $\text{add}(Ae)$ and $\text{add}(Bf)$ are invariant under Nakayama functor, then eAe and fBf are self-injective, and any stable equivalence of Morita type between A and B induces a stable equivalence of Morita type between eAe and fBf . In general, however, our algebras eAe and fBf in Theorem 1.1 may not be self-injective.

As a corollary of Theorem 1.1, we get the following result.

Corollary 2.7 *Suppose A and B be finite-dimensional k -algebras of finite representation type, and let Λ and Γ be the corresponding Auslander algebras of A and B , respectively. Assume that $\Lambda/\text{rad}(\Lambda)$ and $\Gamma/\text{rad}(\Gamma)$ are separable. Then Λ and Γ are stably equivalent of Morita type if and only if A and B are stably equivalent of Morita type.*

Proof. We know that if X is an additive generator for $A\text{-mod}$ with $\Lambda := \text{End}_A(X)$, then $U := \text{Hom}_A(X, D(A_A))$ is a projective-injective Λ -module with $\text{End}_\Lambda(U) \simeq A^{op}$; and every indecomposable projective-injective Λ -module is isomorphic to a direct summand of U . Note that this U satisfies the conditions in Lemma 2.4. If we choose e to be the sum of all idempotents corresponding to the indecomposable injective A -modules, then Lemma 2.6 says that the conditions in Theorem 1.1 on the idempotent $e \in \Lambda$ are satisfied. Note that e defines an idempotent element f in Γ (see Theorem 1.1), and that $\text{add}(\Gamma f)$ contains all projective-injective Γ -modules. With these in mind, the corollary follows from Theorem 1.1 and [11, Theorem 1.1]. \square

For an algebra A , we denote by $[A]$ the class of all those algebras B for which there is a stable equivalence of Morita type between B and A . From the above corollary, we have the following result.

Corollary 2.8 *Suppose that k is a perfect field. Let \mathcal{F} be the set of equivalence classes $[A]$ of representation-finite k -algebras A with respect to stable equivalence of Morita type, and let \mathcal{A} be the set of equivalence classes $[\Lambda]$ of Auslander k -algebras Λ with respect to stable equivalence of Morita type. Then there is an one-to-one correspondence between \mathcal{F} and \mathcal{A} .*

Another consequence of Theorem 1.1 is the following corollary.

Corollary 2.9 *Suppose that A and B are two k -algebras. Let ${}_A X$ be a generator-cogenerator for $A\text{-mod}$ such that $\text{End}_A(X)/\text{rad}(\text{End}_A(X))$ is separable, and let ${}_B Y$ be a generator-cogenerator for $B\text{-mod}$ such that $\text{End}_B(Y)/\text{rad}(\text{End}_B(Y))$ is separable. If $\text{End}_A(X)$ and $\text{End}_B(Y)$ are stably equivalent of Morita type, then so are A and B . In this case, A and B have the same global, dominant, finitistic and representation dimensions.*

Finally, we remark that if we consider derived equivalence instead of stable equivalence of Morita type in Corollary 2.7, then we know from [4] that a derived equivalence between

representation-finite, self-injective algebras A and B implies a derived equivalence between their Auslander algebras. But the converse of this statement is still open. For further information on constructing derived equivalences, we refer the reader to a current paper [5].

3 Higher Auslander algebras

In the following, we point out that Corollary 2.7 holds true for n -representation-finite algebras and n -Auslander algebras studied in [6].

Now we recall some definitions from [6]. Let A be a finite-dimensional k -algebra, and let $n \geq 1$ be a natural number. An A -module T is called an n -cluster tilting module if $\text{add}(T) = \{X \in A\text{-mod} \mid \text{Ext}_A^i(X, T) = 0, 1 \leq i < n\} = \{X \in A\text{-mod} \mid \text{Ext}_A^i(T, X) = 0, 1 \leq i < n\}$. The k -algebra A is called 1-representation-finite if there is an 1-cluster tilting A -module T . This is equivalent to saying that A is representation-finite. For $n \geq 2$, the k -algebra A is called n -representation-finite if $\text{gl.dim}(A) \leq n$ and there is an n -cluster tilting A -module T .

A k -algebra Λ is called an n -Auslander algebra if there is an n -representation-finite k -algebra A with an n -cluster tilting A -module T such that Λ is Morita equivalent to $\text{End}_A(T)$. Note that, for an n -representation-finite algebra A , its n -Auslander algebra is unique up to Morita equivalence.

Clearly, each n -cluster tilting A -module T is a generator and co-generator for $A\text{-mod}$. Thus the indecomposable projective-injective $\text{End}_A(T)$ -modules are of the form $\text{Hom}_A(T, I)$, where I is an indecomposable injective A -module.

Let A be an n -representation-finite k -algebra with T an n -cluster tilting A -module. Furthermore, we assume that A has no separable direct summands and that $A/\text{rad}(A)$ is separable. If A is stably equivalent of Morita type to an algebra B such that B has no separable direct summand and $B/\text{rad}(B)$ is separable, then B is n -representation-finite. In fact, if two indecomposable bimodules ${}_A M_B$ and ${}_B N_A$ define the stable equivalence of Morita type between A and B , then $N \otimes_A T$ is an n -cluster tilting B -module: Since this stable equivalence of Morita type is of adjoint type by Lemma 2.5, we see that $\text{Ext}_A^i(N \otimes_B T, N \otimes_A T) \simeq \text{Ext}_A^i(T, M \otimes_B N \otimes_A T) \simeq \text{Ext}_A^i(T, T \oplus P \otimes_A T) = \text{Ext}_A^i(T, T) \oplus \text{Ext}_A^i(T, P \otimes_A T) = 0$ for $1 \leq i < n$ since $P \otimes_A T$ is a projective-injective A -module. This shows that $\text{add}(N \otimes_A T)$ is contained in both $\{X \in B\text{-mod} \mid \text{Ext}_B^i(X, N \otimes_A T) = 0, 1 \leq i < n\}$ and $\{X \in B\text{-mod} \mid \text{Ext}_B^i(N \otimes_A T, X) = 0, 1 \leq i < n\}$. Now, let $Y \in B\text{-mod}$ such that $\text{Ext}_B^j(N \otimes_A T, Y) = 0$ for $1 \leq j < n$. Then $0 = \text{Ext}_B^j(N \otimes_A T, Y) = \text{Ext}_A^j(T, M \otimes_B Y)$ for $1 \leq j < n$, and therefore $M \otimes_B Y \in \text{add}(T)$. This implies that $Y \in \text{add}(N \otimes_A T)$. Similarly, we show that $\text{add}(N \otimes_A T) = \{Y \in B\text{-mod} \mid \text{Ext}_B^i(Y, N \otimes_A T) = 0, 1 \leq i < n\}$. Hence $N \otimes_A T$ is an n -cluster tilting B -module.

Thus, n -representation-finite k -algebras A with $A/\text{rad}(A)$ separable are closed under stable equivalences of Morita type.

As in the case of Corollary 2.7, the following is a consequence of Theorem 1.1.

Theorem 3.1 *Suppose that A and B are finite-dimensional k -algebras such that both are n -representation-finite. Let Λ and Γ be the corresponding n -Auslander algebras of A and B , respectively. Assume that both $\Lambda/\text{rad}(\Lambda)$ and $\Gamma/\text{rad}(\Gamma)$ are separable. Then Λ and Γ are stably equivalent of Morita type if and only if A and B are stably equivalent of Morita type.*

Proof. For $n = 1$, we have done by Corollary 2.7. Let $n \geq 2$. Suppose ${}_A T$ is an n -cluster tilting A -module such that $\text{End}_A(T) = \Lambda$, and suppose ${}_B S$ is an n -cluster tilting B -module such that $\text{End}_B(S) = \Gamma$. If ${}_A M_B$ and ${}_B N_A$ are two indecomposable bimodules defining a stable equivalence of Morita type between A and B , then, by the above discussion, we know that Γ is Morita equivalent to $\text{End}_B(N \otimes_A T)$. Now, we use [11, Theorem 1.1, or Theorem 1.3] which states that if R is an A -module with $\text{add}({}_A A) \subseteq \text{add}(R)$, then $\text{End}_A(R)$ and $\text{End}_B(N \otimes_A R)$ are stably equivalent of Morita type.

Conversely, suppose that two bimodules ${}_A X_\Gamma$ and ${}_B Y_\Lambda$ define a stable equivalence of Morita type between Λ and Γ . Note that $\text{add}(T)$ contains both $\text{add}({}_A A)$ and $\text{add}(D({}_A A))$. Let e be an

idempotent in Λ such that $\text{add}(\Lambda e)$ is just the category of projective-injective Λ -modules. Then $e\Lambda e$ is Morita equivalent to A^{op} . As in Theorem 1.1, we have an idempotent f in Γ such that $f\Gamma f$ is Morita equivalent to B^{op} . Thus a stable equivalence of Morita type between Λ and Γ implies a stable equivalence of Morita type between A^{op} and B^{op} by Theorem 1.1, and therefore a stable equivalence of Morita type between A and B . \square

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May 20, 2009